MA 161: Lesson 31

Approximating area under curves (5.1)

Riemann Sums

Left and Right sums

Summation Notation

Coming up Next: Definite Integrals

Announcements

Exam 3 on Nov 20th - Lesson 18 (3.10) to Lesson 30 (4.9)

Study guide and instructions for exam 3 posted on brightspace

Feasting with faculty - tomorrow 11:15am -12:15pm (Windsor)

Office Hours

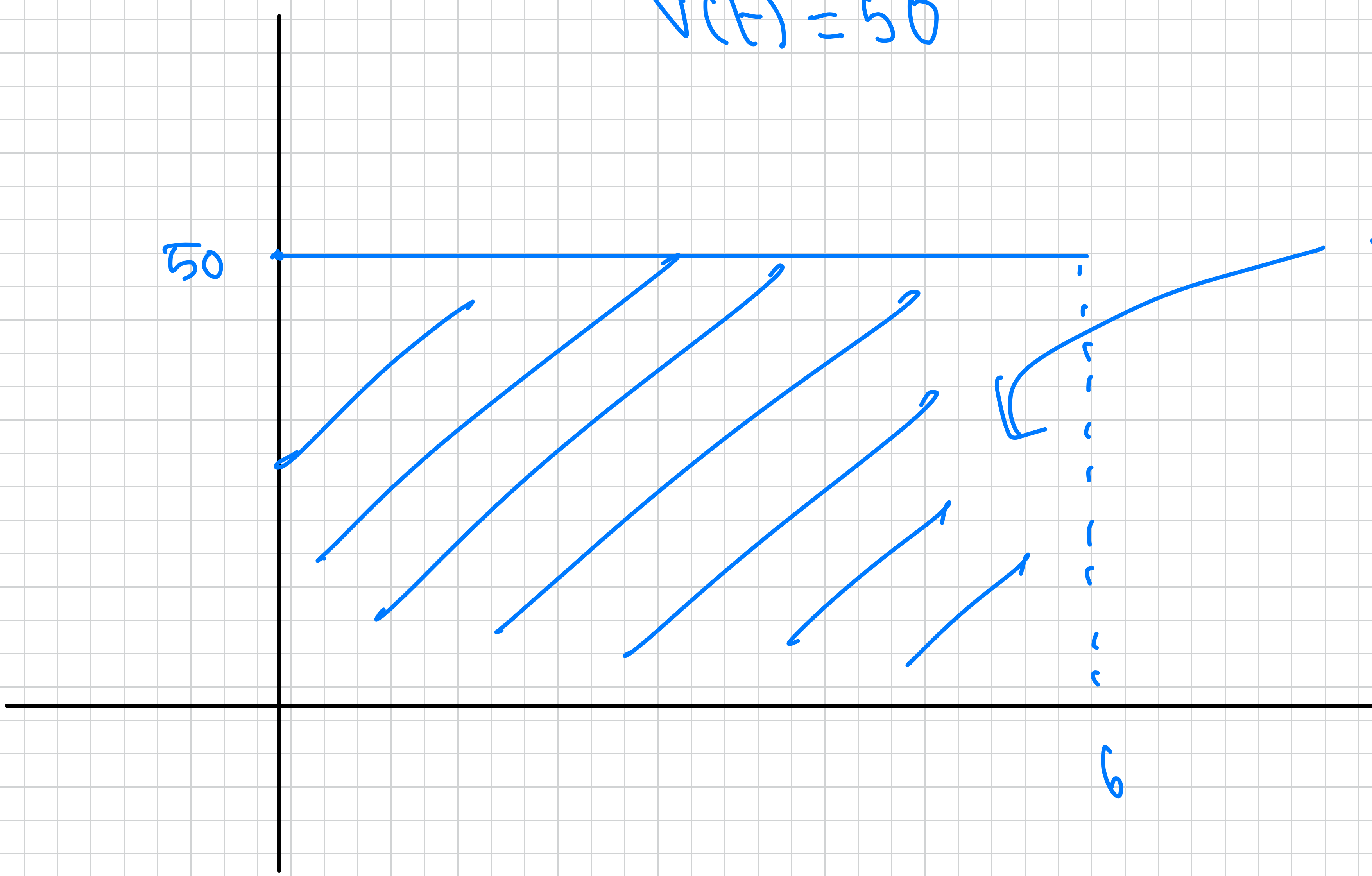
M, W, F: 245pm - 415pm

Warmup Example:

Suppose you are driving on a straight road for 6 hours at a constant speed of 50 mi/hr. How far are you from the start?

$\rightsquigarrow$ $50 \times 6 = 300$ miles

$V(t) = 50$



Area = 300

$V(t)$ = derivative of $\underbrace{S(t)}_{\text{position}}$

$S(t)$ = Antiderivative of $V(t)$

$= 50t + C$

e.g: Find Area under $f(x) = -32x + 48$ on $[0, 3/2]$

Area = Area of triangle

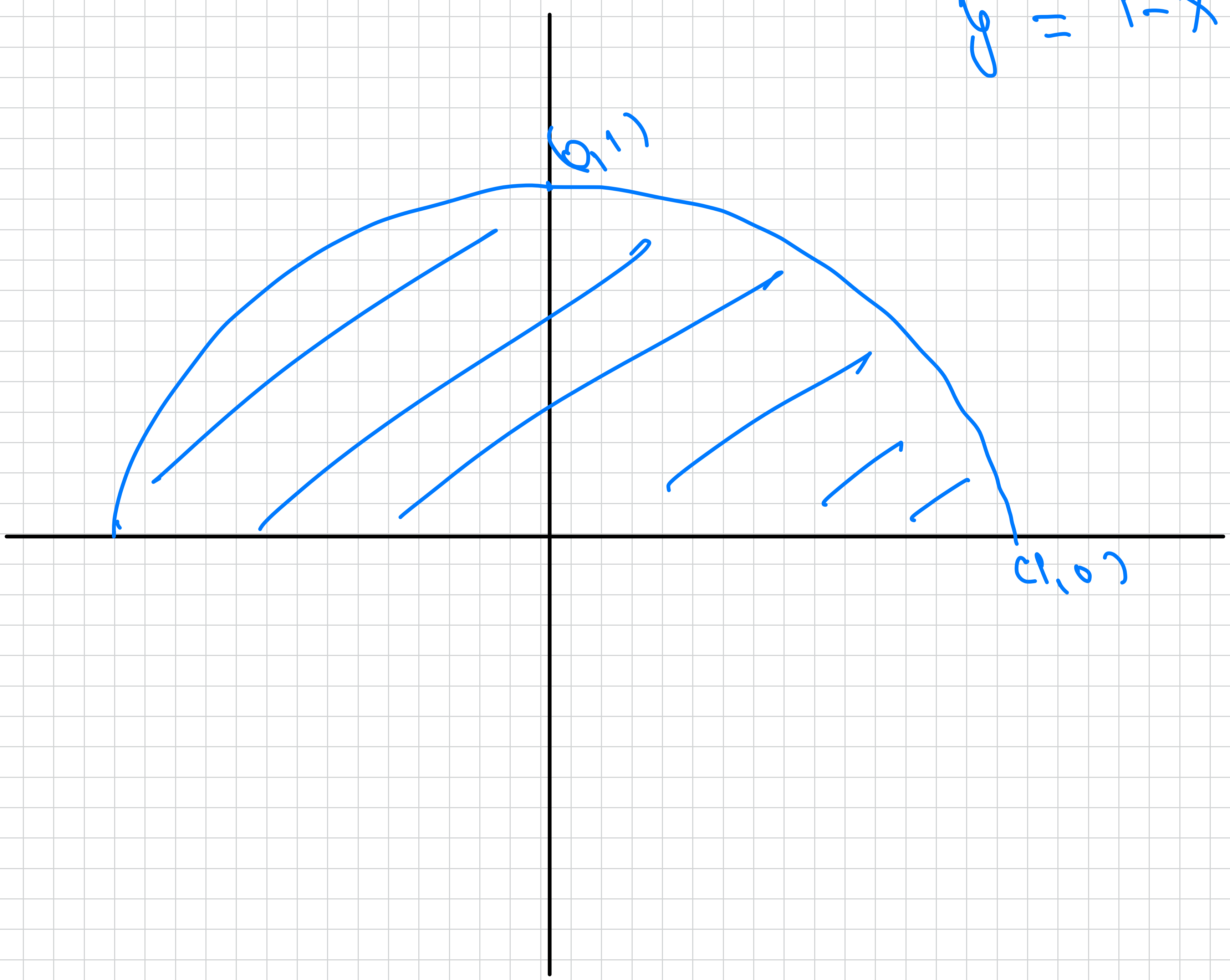
$= \frac{1}{2} * base * height$

$= \frac{1}{2} * \frac{3}{2} * 48$

$= 36.$

e.g find Area under $y = \sqrt{1-x^2}$ on $[-1, 1]$

$$y^2 = 1 - x^2 \rightarrow x^2 + y^2 = 1$$



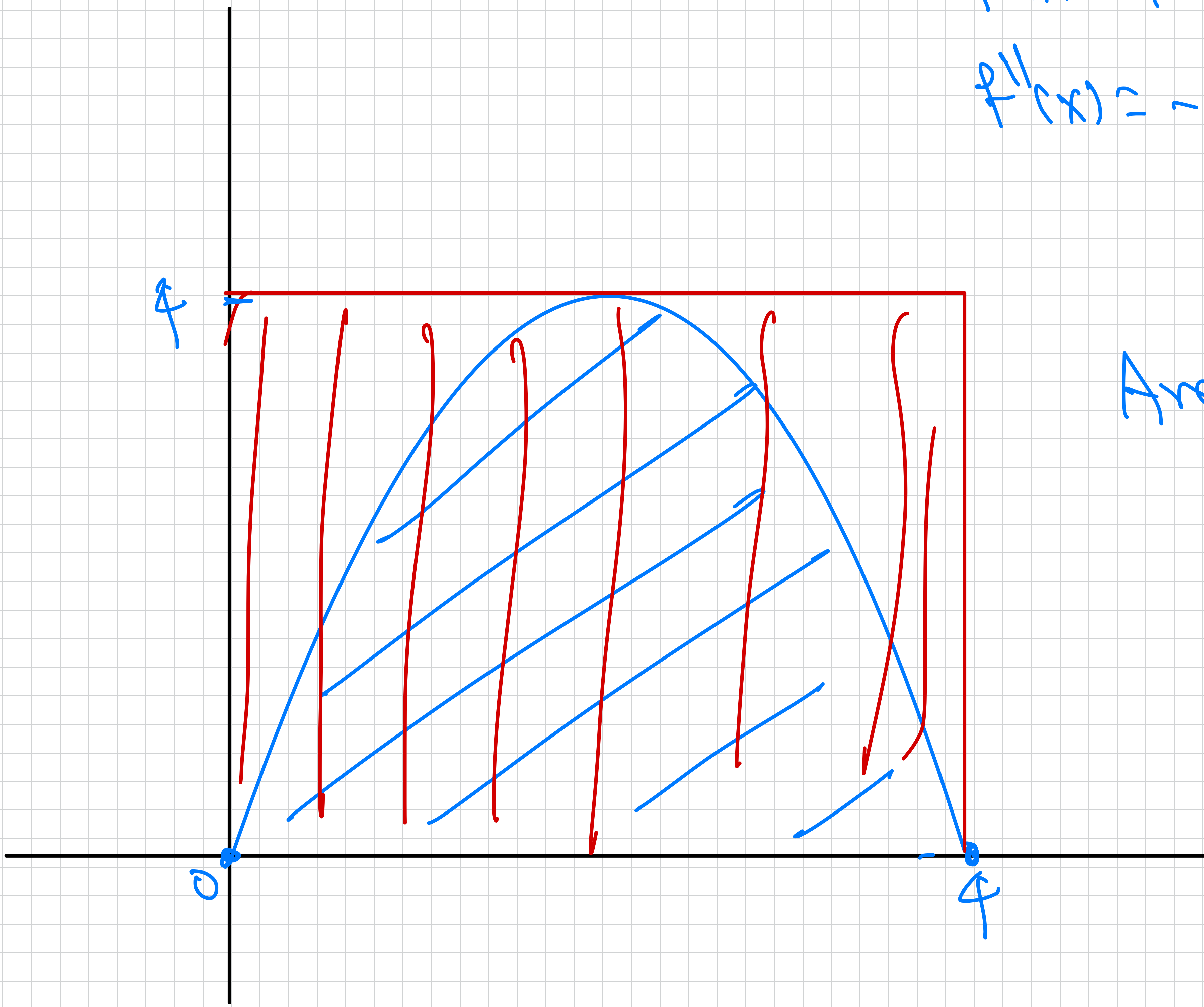
Area = Area of semi circle of Radius 1

$$= \frac{1}{2} * \pi * 1^2 = \frac{\pi}{2}$$

eg: Find Area under the Curve $f(x) = 4x - x^2$ on $[0,4]$

$f'(x) = 4 - 2x = 0 \rightsquigarrow x = 2$ critical

$f''(x) = -2 < 0 \rightsquigarrow$ c. down



Area cannot be computed using formulas we know.

$0 < \text{Area} < 16$

idea: Divide into small rectangles and approximate Area

4- Rectangle(s)

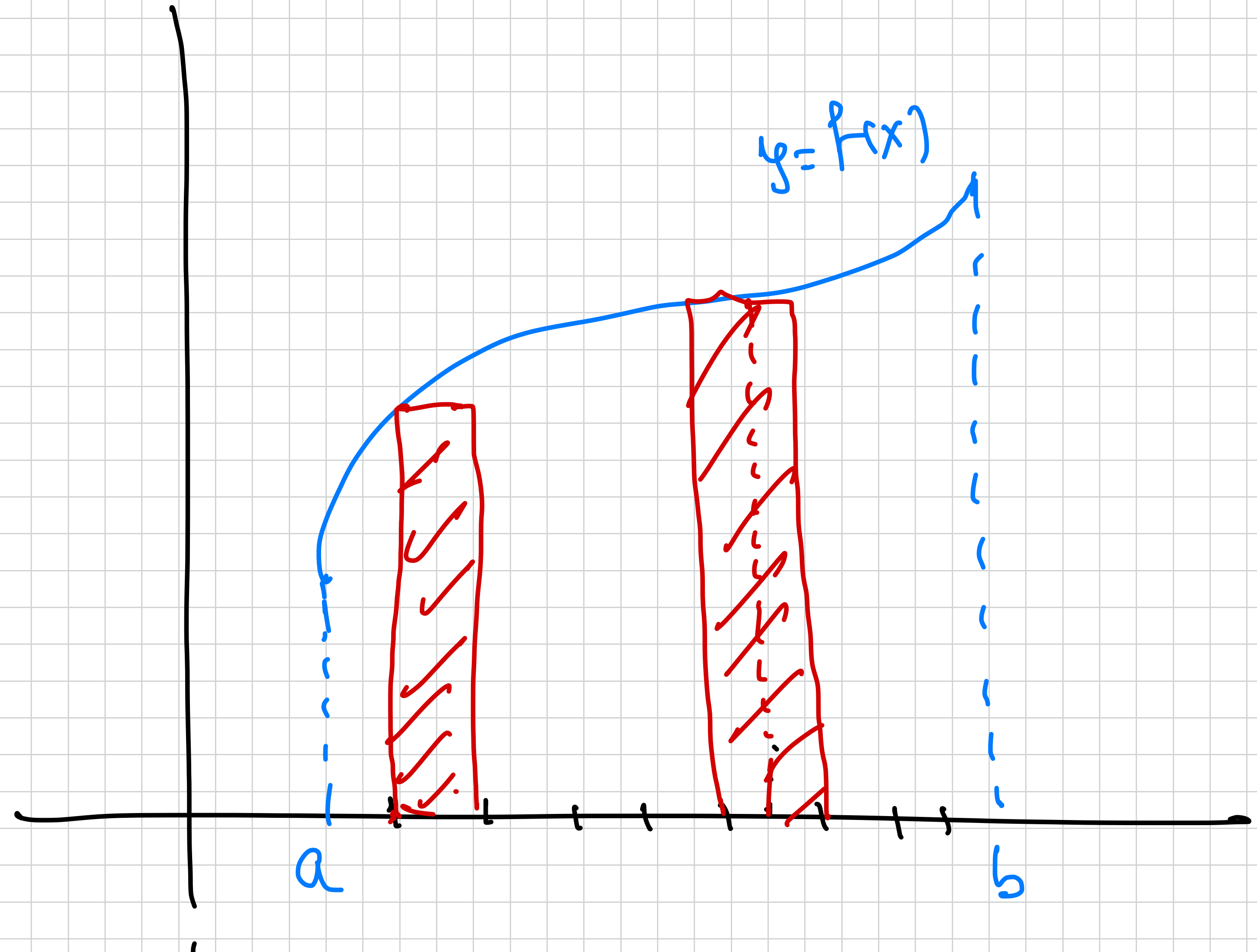
Using 8- Rectangles

Riemann Sum $\rightsquigarrow$ Approximates area under $y=f(x)$ on $[a,b]$ using Rectangles

① Divide $[a,b]$ into n equal parts

each has width $\Delta x = \frac{b-a}{n}$

② in each small interval choose a sample point x_k & form a Rectangle with height $f(x_k)$

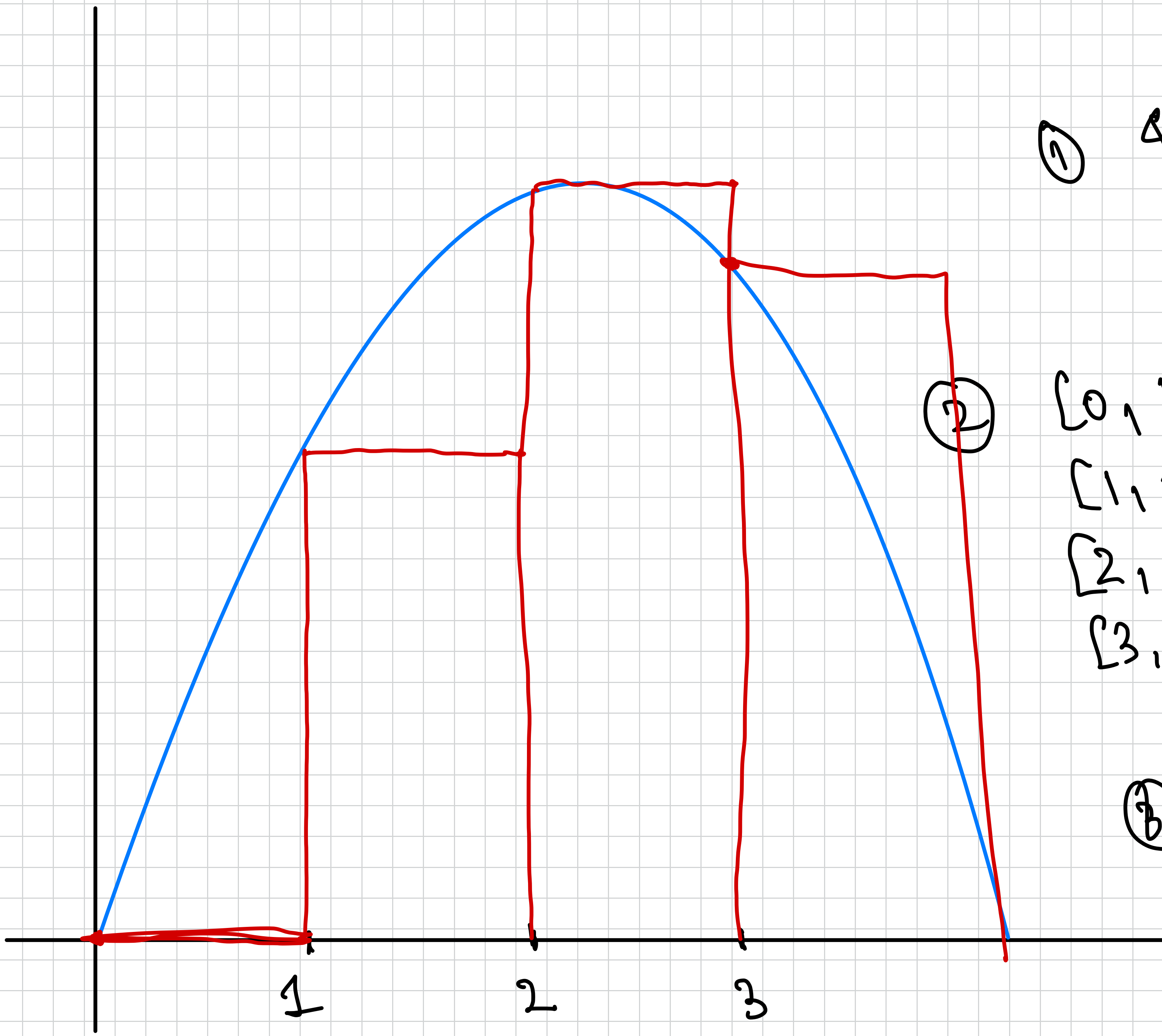


③ Approximate Area by Adding Areas of all the rectangles

Approx. Area using n intervals $= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Approximate Area under $f(x) = 4x - x^2$ on $[0,4]$ using 4 Rectangles and left end points.

Notation: L_4



① 4 equal interval of $[0,4]$

width $= \frac{4-0}{4} = 1$

②
$$\left.\begin{array}{l} [0,1] \rightsquigarrow x_1 = 0 \\ [1,2] \rightsquigarrow x_2 = 1 \\ [2,3] \rightsquigarrow x_3 = 2 \\ [3,4] \rightsquigarrow x_4 = 3 \end{array}\right\} \text{heights} \quad \begin{array}{l} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{array}$$

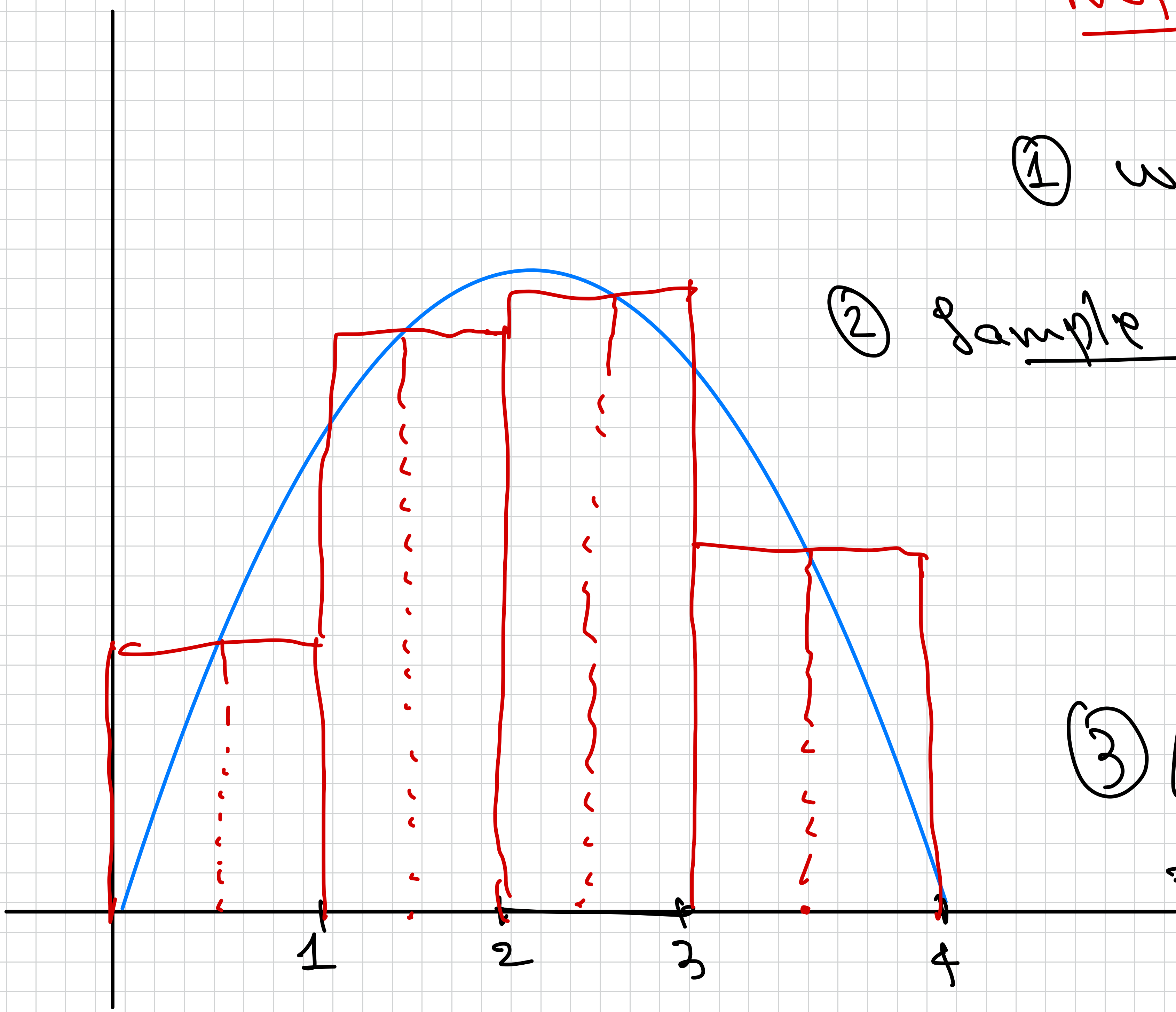
③ Approx Area $= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

$= f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1$

$= 0 + 3 + 4 + 3 = \underline{10}$

Approximate Area under $f(x) = 4x - x^2$ on $[0, 4]$ using 4 Rectangles and mid points

Notation: M_4



① width $= \Delta x = \frac{4-0}{4} = 1$

② sample points:

$x_1 = 0.5$
$x_2 = 1.5$
$x_3 = 2.5$
$x_4 = 3.5$

heights

$f(x_1)$
$f(x_2)$
$f(x_3)$
$f(x_4)$

③ Approx Area:

$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

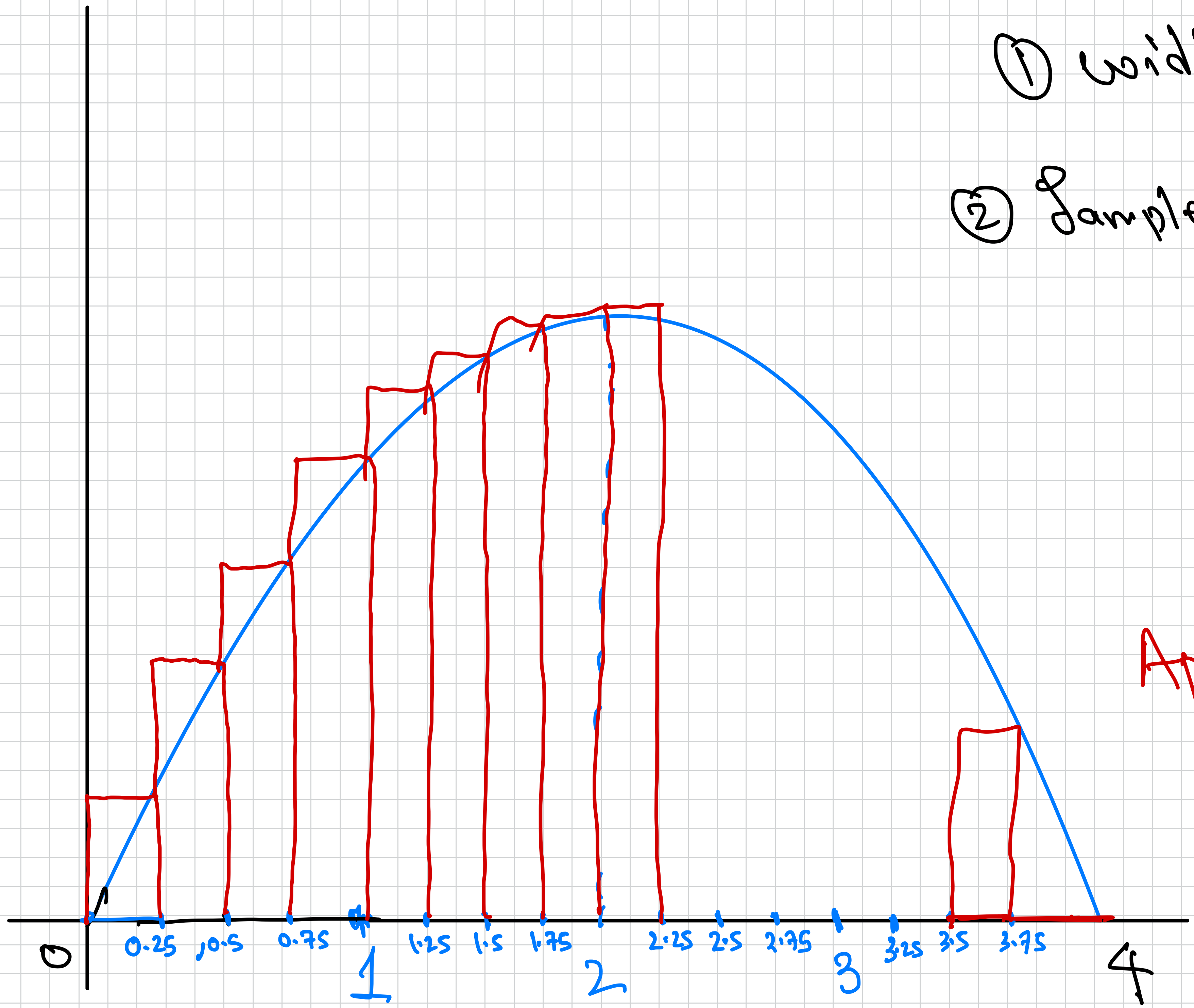
$= f(0.5) + f(1.5) + f(2.5) + f(3.5)$

$= 1.75 + 3.75 + 3.75 + 1.75 = 11.$

Actual Area = 10.667 (Next class)

divide into smaller Rectangles $\rightsquigarrow$ get Better Approx.

eg: 16 Rectangles, Right end point $\rightsquigarrow$ R_{16}



① width $= \Delta x = \frac{4}{16} = \frac{1}{4} = 0.25$

② Sample points: $x_1 = 0.25$

$x_2 = 0.5$

$x_3 = 0.75$

$\vdots$

$x_{15} = 3.75$

$x_{16} = 4$

Approx Area

$= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{16})\Delta x$

$= \underline{10.625}$

$$\sum_{k=1}^{16} f(x_k)\,\Delta x$$

Summation Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Using this notation: nth Rieman sum of $f(x)$ on $[a,b]$

① width $\Delta x = \frac{b-a}{n}$

② Sample points $x_1, \ldots, x_n$

$$\text{Approx. Area} = \sum_{k=1}^{n} f(x_k)\,\Delta x$$

Some important Summation Formulas

① $$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{5} k = 1+2+3+4+5 = 5(6)/2 = 15$$

$$1+2+\cdots+10 = \frac{10(11)}{2} = 55$$

② $$\sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1+2^2+3^2 = \frac{3(3+1)(7)}{6} = 14$$

③ $$\sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^3+2^3+3^3 = \frac{3^2\,4^2}{4} = \underline{\underline{36.}}$$